PERGAMON

International Journal of Solids and Structures 36 (1999) 5207–5232

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

On a viscoplastic Shanley-like model under constant load

Andrea Benedetti[a,]*, Luca Deseri[b]

[a] *DISTART, Facoltà di Ingegneria, Università di Bologna, viale Risorgimento 2, 40134 Bologna, Italy*
[b] *Dipartimento di Ingegneria, Università di Ferrara, via Saragat 1, 44100 Ferrara, Italy*

Received 24 November 1997; in revised form 19 June 1998

Abstract

Motivated by applications devoted to study the behavior of steel and aluminum alloys columns, inelastic Shanley-like models have been extensively studied in the literature, mainly to investigate buckling and post buckling problems (see Sewell, 1971; Hutchinson, 1974 for a complete review).

On the other hand, recent papers discussing geotechnical problems point out that those models may be useful for the study of the essential features of the equilibrium of towers. In this case, the structure's proper weight (which is a conservative load with constant magnitude), and the verticality imperfection, appear to be responsible for the leaning evolution, as well as the time variation of the mechanical property of the soil.

Throughout this paper, a 'T' shaped rigid rod on two no-tension viscoplastic springs under constant load with initial imperfection is considered. Under fairly general assumptions, a viscoplastic constitutive law is derived as a particular case of the theory developed in (Gurtin et al., 1980), studying its behavior under loading processes. By virtue of a time rescaling procedure, extreme retardation leads to determine a yielding parameter, which allows to distinguish between viscoelastic and viscoplastic ranges.

For all the states attained by the rod, explicit expressions for the two displacement parameters characterizing its evolution are given. Noting that failure may occur if the reaction of one spring goes to zero, sufficient conditions under which no bifurcation and no failure occur are given for all the phases, leading so to determine the minimum upper bound for the load parameter. This new result turns out to depend only on the relaxation surface parameters at equilibrium, irrespective of the behavior under non-zero finite deformation velocities.

1. Introduction

Studies about discrete and continuum Shanley-like models had a great development from the late 40s up to 60s (Shanley, 1947; Hill, 1960; Hoff, 1954, 1956, 1958; Libove, 1952; Rabotnov and Shesterikov, 1957; Duberg, 1962; Kempner, 1962; Sewell, 1971 and the references cited therein), motivated by the analysis of the inelastic buckling of steel and aluminum alloy columns, mainly for aeronautical and mechanical applications.

* Corresponding author

PII: S0020-7683(98)00195-4

The first studies about this problem started at the end of the last century (Engesser, 1889; Considère, 1891), but Von Kármán (1910) gave the first rational derivation of an estimate of the bifurcation load. In this case, considering elastoplastic-hardening behavior of the material, and denoting by K and H the moduli in the elastic and plastic range, Von Kármán found that the critical load depends on the reduced modulus $2KH/(K+H)$. In the early 20s, on the basis of experimental observations (Basquin, 1924; Duberg, 1962) it was recognized that a more reliable prediction of the bifurcation value of the load could be obtained by replacing the reduced modulus by the hardening (or tangent) modulus H. This result was in agreement with the Engesser's conjecture made in 1889. The explanation of this fact was given in a paper of Shanley (1947), where it is shown that the hidden assumption present in the Von Kármán's analysis is that the load was considered constant. More recently, for Shanley-like elastoplastic models, post-buckling analyses have been worked out by many authors too (see Hutchinson, 1974; Triantafylldis, 1983; Cimetière and Léger, 1996 among others).

However, in order to refine the predictions obtained by elastoplastic constitutive equations, starting from the 50s viscoelastic and viscoplastic buckling of columns have been extensively investigated (Hoff, 1954, 1956, 1958; Kempner, 1962; Rabotnov, 1969; Bazant and Cedolin, 1991 and the references cited therein). These studies have essentially been carried out starting from different incremental creep power laws, and their main result is the statement of the so-called critical time of the system. This concept was associated either to the time needed to reach a bifurcation point, or to the time resulting in an infinite deflection (Rabotnov and Shesterikov, 1957; Rabotnov, 1969), although geometrical non-linearity was disregarded. In Vinogradov (1985) these effects were taken into account, despite only a linear viscoelastic constitutive law was considered, resulting in an approximate treatment of the convolution integral operator.

Besides the analyses cited above, many models and theories have been developed in viscoplasticity (Chaboche and Rousselier, 1983; Chaboche, 1993; Cernocky and Krempl, 1979; Cristescu and Suliciu, 1982; Gurtin et al., 1980; Haupt, 1992, 1993; Kratochvil and Dillon, 1969; Krempl, 1975; Lubliner, 1973; Malinin and Kandjinsky, 1972; Naghdi and Murch, 1963; Oka et al., 1988; Odqvist, 1966; Perzyna, 1963, 1966; Rzhanitsyn, 1968; Tsamakis, 1996), but, contrary to the viscoelasticity, where the unifying effort has led to a general formulation (Coleman and Noll, 1961; Gurtin and Sternberg, 1962; Day, 1972; Dill, 1975; Fabrizio et al., 1994; Del Piero and Deseri, 1997), without leading to a unified theory.

For the reasons outlined above, it is understandable why, even for simple models like that examined in this paper, a fully coherent approach to the viscoplastic equilibrium problem has not yet been established. The difficulties to overcome can be summarized as follows:

(a) in performing buckling analyses the variation of the load's magnitude should be taken into account;
(b) many theories of viscoplasticity can be likewise considered to hold;
(c) 'small' geometrical non-linearity is sufficiently accurate to bring up bifurcation points of the system (the meaning of the attribute 'small' will be clarified in the next section). However, it is worth noting that this assumption is not compatible with the traditional concept of critical time.

Although the study of buckling was crucial for problems related to aeronautical applications, where the statement (a) plays a key role, there are problems in geotechnical engineering in which the variation of the magnitude of the external actions is not so meaningful. This circumstance occurs for the equilibrium problem of leaning towers (Burland and Viggiani, 1994; Hambly, 1985; Heyman, 1992; Lancellotta, 1993), where the self weight of the construction and the initial imperfection of verticality appear to be the most important variables driving the evolution of the leaning. Moreover, the time evolution of the mechanical properties of the soil is a crucial point to consider for an accurate description of the phenomenon (Nova and Montrasio, 1995; Cheney et al., 1991).

Then, a viscoplastic Shanley-like model under constant load seems to be the simpler one capable to point out the essential features of this problem. In the paper, a 'T' shaped 'slender' rigid rod simply supported by two springs having such a type of constitutive law is considered, where the meaning of 'slender', the simple kinematics and the equilibrium problem are outlined in Section 2. Moreover, a conservative constant external load is considered applied in the center of gravity of the system, and an initial imperfection is assumed to be present starting from the beginning of the leaning evolution.

The constitutive law is introduced in Section 3. Among the theories developed for viscoplasticity, the one formerly introduced by Lubliner (1973) in a more general context, is considered here in its one-dimensional form stated by Gurtin et al. (1980): in this theory the stress evolution equation may be governed by a piecewise smooth relaxation surface in the state space, so that both the viscoelastic and viscoplastic ranges are described by this constitutive law. A piecewise quasi-linear form of that surface is constructed in Section 3, to get a model which reproduces the property of rate sensitivity of the yielding threshold. Loading processes permit to use the constitutive law in its integral form, in terms of an appropriate choice of the stress–strain measures. A real parameter associated to the yielding can be determined by the condition that the extreme retardation of the stress of the more compressed spring becomes equal to the equilibrium yielding stress.

In Section 4, the evolution of the system is studied starting from the viscoelastic behavior. A concept of failure, associated with the ultimate rotation achievable by the system when one spring is totally unloaded, is introduced. The assumption related to this concept is that no-tension is supported by the springs, as it is rather common when unilateral soil–structure interaction is considered. It is worth noting that the investigation of the extreme equilibrium conditions leading to the failure of the system has the same importance of the characterization of the bifurcation. Values of the load parameter at bifurcation and failure are computed in the viscoelastic phase, and a sufficient condition under which the listed phenomena do not arise is given. If the external load is lower than the minimum between those values, the system can enter the viscoplastic phase, and we call such a load admissible.

Section 5 is devoted first of all to study the yielding occurrence. The evaluation of the yielding parameter allows not only to calculate explicitly the effective yielding value of the reaction in the more compressed spring, but also to set a bilateral delimitation of it, which does not depend on the yielding parameter itself.

In the same section, the equilibrium equations are studied making use of the Laplace-transform technique. By this way, it is possible to have the explicit expression of the solution (see Appendix C), and to characterize directly the asymptotic behavior of the system. This is worked out in Section 6, where, first of all, a sufficient condition for the uniqueness of the solution is given in terms of the Von Kármán critical load associated to the reduced equilibrium modulus. Moreover, it is shown that this value of the load parameter does not guarantee whether the failure of the system could occur. As a consequence, the ultimate value of the rotation can be attained in a finite time only if the external load is admissible and lower than the Von Kármán bifurcation value. In this sense, admissible values of the load for which failure does not arise neither in a finite time (which will be called critical time), nor asymptotically, will be defined attainable.

Finally, a sufficient condition of attainability is given. The load parameter which appears in this condition is given by the sum of the Shanley critical load, associated to the equilibrium hardening modulus, and a positive term which depends linearly on the equilibrium yielding stress and on the module's ratio in the viscoelastic and viscoplastic phases. For this reason the above-mentioned results are fairly general, as they do not depend on the particular expression for the relaxation surface appearing in the constitutive equation.

2. Compatibility and equilibrium equations

Here we consider a slender 'T' shaped rigid-rod (Fig. 1). supported by two no-tension springs having a viscoelasto–plastic hardening behavior, described by the constitutive equation discussed in Section 3. The attribute slender means that the following assumption is verified:

$$\phi_L \ll 1, \tag{I}$$

where the parameter ϕ_L is defined as follows:

$$\phi_L := \frac{B}{2h_G}. \tag{2.1}$$

In particular, we take into account the initial out of verticality $\phi_0 > 0$ of the rod axis, which is assumed small in the sense that the following inequality holds:

$$\frac{\phi_0}{\phi_L} \ll 1. \tag{II}$$

A more accurate analysis which takes into account exact geometrical non-linearity according to Cimetière and Léger (1996) may be done removing assumption (I) and replacing $\ll$ by $<$ in (II). We denote with $\phi(t)$ the present value of the additional rotation, and a part from the total rotation $\phi_0+\phi(t)$, the configuration of the system is uniquely determined by the knowledge of the present value $u(t)$ of the vertical displacement of the base middle point. The simple kinematics of the system can be described by the compatibility equations for the springs:

$$E_1(t) = u(t) - \frac{B}{2}\phi(t), \tag{2.2}$$

$$E_2(t) = u(t) + \frac{B}{2}\phi(t). \tag{2.3}$$

If N_1 and N_2 are the reactions in the springs, positive if in compression, by assumption (I) the linear approximation of the geometrical effects can be taken into account, so that the equilibrium equations become:

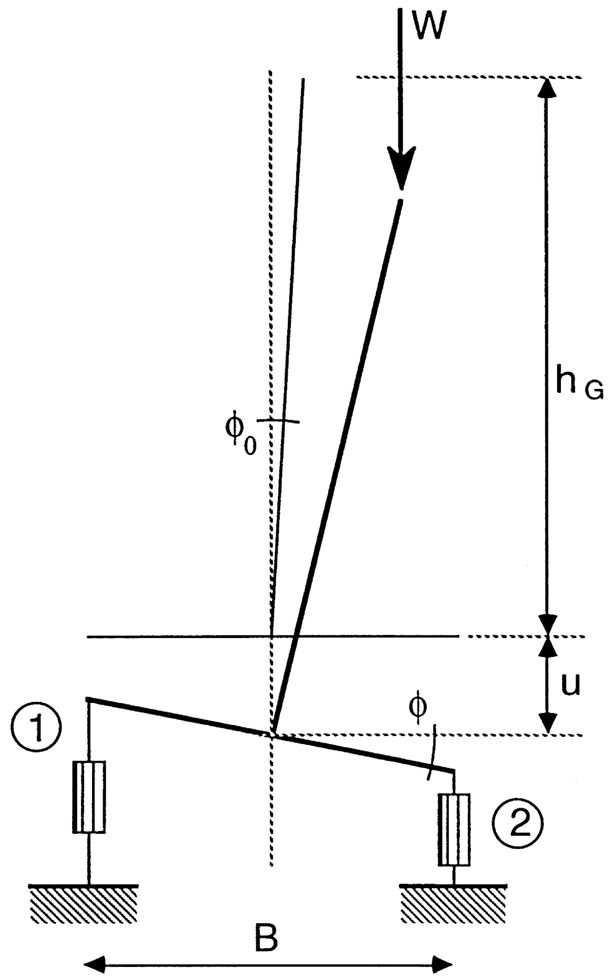

Fig. 1. Geometry and data of the 'T' shaped Shanley column with viscoplastic springs.

$$N_1(t)+N_2(t) = W$$

$$\frac{B}{2}[-N_1(t)+N_2(t)] = Wh_G[\phi_0+\phi(t)], \tag{2.4}$$

leading to the following expressions:

$$N_1(t) = \frac{W}{2}\left\{1-\frac{2h_G}{B}[\phi_0+\phi(t)]\right\}, \tag{2.5}$$

$$N_2(t) = \frac{W}{2}\left\{1+\frac{2h_G}{B}[\phi_0+\phi(t)]\right\}. \tag{2.6}$$

It is worth remarking that if $t \mapsto \phi(t)$ is positive for every t, by eqn (2.6), we have

$$N_2(t) \geqslant \frac{W}{2}. \tag{2.7}$$

Moreover, if $t \mapsto \phi(t)$ is also monotonic increasing, the function $t \mapsto N_2(t)$ has the same behavior and this ensures the impossibility of unloading after yielding.

Furthermore, since no-tension springs are considered, we say that the ultimate limit equilibrium condition of the system occurs if and only if the less compressed spring reaches the null value. Using (2.1) and the first of (2.4), this condition allows us to define the ultimate rotation ϕ_u as

$$\phi_0 + \phi_u := \phi_L \tag{2.8}$$

By eqn (2.5), the monotonicity of $t \mapsto \phi(t)$ implies that $t \mapsto N_1(t)$ is monotonic decreasing, thus this function does not attain the null value during the evolution of the system if and only if

$$N_2(t) \leqslant W \tag{2.9}$$

for every t. If the ultimate limit equilibrium condition is reached, (2.9) is verified as an equality, and this would arise either at a finite instant of time, and we say that there exists a critical time of failure for the system, or asymptotically in the limit when t goes to infinity. In the traditional literature, the notion of critical time is associated with the occurrence of infinite deflection of a system, even if exact geometrical non-linearity is left out. More precisely, in the present context the critical time is related to the loss of equilibrium of the system itself, which is allowed to reach the ultimate rotation defined above. The asymptotic failure condition can be equivalently investigated by comparison between $\phi(\infty)$ and ϕ_u, being $\phi(\infty)$ the finite asymptotic value of $t \mapsto \phi(t)$, which does exist because eqn (2.6) holds, and $N_2(\infty) \leqslant W$ according to (2.9). If this is the case, it is possible to compute the value $\bar{W}_U^{\infty}(\phi_0)$ of the external load associated to the asymptotic failure of the system.

On the other hand, though the ultimate equilibrium condition must be avoided, the external load has also to be small enough to guarantee that no equilibrium bifurcation phenomena occur during the evolution. Thus, the minimum upper-bound $\bar{W}$ of the external load has to be determined in such a way that if the following inequality is satisfied

$$0 < W < \bar{W}, \tag{III}$$

the uniqueness of the equilibrated configuration is ensured for any $t > 0$ and no failure occurs.

3. Constitutive equation

In the present section we consider a constitutive equation for materials with viscoelastic and viscoplastic ranges. If with $E(t)$ we denote the present value of the deformation, and with $s \mapsto E^t(s) := E(t-s)$, $0 < s < +\infty$, the deformation history up to the time t, we assume that even in the viscoplastic range the stress depends linearly on both the entries of the pair $(E(t), E^t(\cdot))$. The response functional of the one-dimensional linear viscoelastic material is given by the classical Boltzmann–Volterra equation

$$N(t) = K \cdot E(t) + \int_0^{+\infty} \dot{G}(s) \cdot E^t(s)\,\mathrm{d}s, \tag{3.1}$$

where K is the instantaneous elastic modulus, and $s \mapsto \dot{G}(s)$ is the derivative of the relaxation function of the material. In a recent paper (Del Piero and Deseri, 1995) some a priori restrictions on G have been deduced in the three-dimensional case, as consequences of some properties of the work done on particular classes of processes, by assuming $s \mapsto E^t(s)$, $0 < s < +\infty$, of bounded variation for any t, and $\dot{G}$ integrable.

The particularization of some of the previously cited restrictions to the one-dimensional case leads to the inequalities:

$$K_\infty > 0; \tag{IV}$$

$$K - K_\infty > 0. \tag{V}$$

In the following we will consider G of exponential type, i.e.

$$G(t-r) = K_\infty + \bar{K}\mathrm{e}^{-(t-r)\alpha}, \tag{3.2}$$

$-\infty < r < t$, where $\bar{K} := K - K_\infty$ and α is a positive real.

In Del Piero and Deseri (1997), the concepts of states and processes for the three-dimensional linear viscoelastic materials have been properly introduced and discussed. In particular, it has been shown that the state of linear viscoelastic materials characterized by relaxation functions of exponential type is determined by the knowledge of the pair $\{N(t), E(t)\}$, formed by the present values of the stress and of the deformation. In other words, as it is well known, linear viscoelastic materials of integral type with exponential relaxation function can be viewed as rate-type linear viscoelastic ones (Gurtin et al., 1980). By substitution of (3.2) into (3.1), a differentiation with respect to t of eqn (3.2) leads to the following equation:

$$\dot{N}(t) - K\dot{E}(t) = \mathscr{G}_a(N(t), E(t)), \tag{3.3}$$

after setting:

$$\mathscr{G}_a(N(t), E(t)) := -\alpha(N(t) - K_\infty E(t)). \tag{3.4}$$

In the model $\mathscr{G}_a$ is the so-called relaxation surface and it is defined in the state's space. Lubliner formerly introduced this concept in 1973 in a more general context, though this denomination has been introduced in Gurtin et al. (1980). It is worth remarking that in Del Piero and Deseri (1995, Section 7) it has been shown that the function $t \mapsto N(t) - K \cdot E(t)$ is differentiable even in the case in which $s \mapsto E^t(s)$, $0 < s < +\infty$, is a function of bounded variation for every t.

Many constitutive models have been presented in the literature, concerning viscoplasticity (Perzyna, 1963; Lubliner, 1973; Gurtin et al., 1980); in particular, as Lubliner noted in 1973, the notion of relaxation surface overcomes the introduction of the concept of yield surface, which is typical of classical plasticity. In fact, the existence of that surface is not needed, because an appropriate choice of the state's functions appearing in the evolution equation takes into account any yielding effect. Moreover, according to the theory mentioned above, if the relaxation surface is a smooth function as in (3.4), eqn (3.3) outlines a viscoelastic behavior; on the other hand, if that surface is piecewise smooth, (3.3) can reproduce viscoplastic behavior. In order to characterize loading and unloading processes, a scalar-valued linear function of the deformation velocity must be introduced (Lubliner, 1973) and in the present context, this function is coincident with the time derivative of the stress. Explicit examples which fit with the classical approach to viscoplasticity, and with the one discussed in Haupt (1992), are given in Cernocky and Krempl (1979), Tsakmakis (1996).

Following the scheme outlined above, we introduce a constitutive parameter N_{ye} that represents the equilibrium yielding threshold, and new measures for the stress and strain:

$$S(t) := |N(t)| - N_{ye}, \tag{3.5}$$

$$D(t) := |E(t)| - K_{\infty}^{-1} N_{ye}. \tag{3.6}$$

If we denote with $N_y > 0$ the yielding threshold of the material subjected to a given history, the corresponding modified stress can be defined as

$$S_y := N_y - N_{ye}, \tag{3.7}$$

so that the stress part exceeding the yielding value holds

$$S^*(t) := S(t) - S_y. \tag{3.8}$$

It is worth noting that N_y is not prescribed, and the relationship between N_{ye} and N_y must be determined: indeed this does depend on the past history and on the present value of the deformation. In analogy with eqn (3.3), the governing equation for evolution for the new measure of the stress takes the form

$$\dot{S}(t) - K\dot{D}(t) = \mathscr{G}(S(t), D(t)), \tag{3.9}$$

where

$$\mathscr{G}(S(t), D(t)) := \begin{cases} \mathscr{G}_a(S(t), D(t)) & \text{for } S^*(t) < 0, \quad \text{or } S^*(t) \geqslant 0 \quad \text{and} \quad \dot{S}(t) < 0 \\ \mathscr{G}_b(S(t), D(t)) & \text{for } S^*(t) \geqslant 0, \quad \text{and } \dot{S}(t) \geqslant 0 \end{cases}. \tag{3.10}$$

The plastic part of the relaxation surface is defined as follows:

$$\mathscr{G}_b(S(t), D(t)) := -\beta(S(t) - H_{\infty} D(t)) - \mathscr{L}(S_y, D_y, t), \tag{3.11}$$

where $\mathscr{L}$ is a function whose explicit expression will be given in Section 5 such that its limit when the velocity goes to zero is zero. Moreover, β and H_{∞} are positive constants such that:

$$\begin{aligned} K - K_{\infty} > K_{\infty} - H_{\infty} > 0, \\ \alpha > \beta, \\ H_{\infty} > 0. \end{aligned} \tag{VI}$$

In order to investigate the properties of the constitutive eqn (3.9), we can follow a time rescaling procedure, as discussed in Del Piero and Deseri (1995, Section 3). Using the monotonicity property of this procedure we can order processes with different velocity and assess the limit when this last goes to zero (see Appendix A). In this case by (3.4), (3.10), and (3.11), we obtain:

$$0 = \begin{cases} S(t) - K_\infty D(t) & \text{for } S(t) < 0, \quad \text{or } S(t) \geqslant 0 \quad \text{and} \quad \dot{S}(t) \leqslant 0 \\ S(t) - H_\infty D(t) & \text{for } S(t) \geqslant 0 \quad \text{and } \dot{S}(t) > 0 \end{cases}. \tag{3.12}$$

This equation describes a piecewise linear path in the space of the states, which is crossed only in the limit for null velocity and for this reason it is called equilibrium path. It is worth noting that the variable t assumes no longer the meaning of time but that of a real parameter driving a classical elastoplastic behavior. The value $t = t_y$ in which $N_y = N_{ye}$, i.e. $S_y = S(t_y) = 0$ in (3.12), can be defined solving $|N(t_y)| = N_{ye}$ when N belongs to the locus described by (3.12).

Corresponding to N_{ye}, the values of the deformation restricted to the same locus hold:

$$E(t_y) = \pm K_\infty^{-1} N_{ye}. \tag{3.13}$$

Further, if we consider the time rescaling defining the map $\gamma \mapsto N_\gamma(t)$ as in eqn (A5) of Appendix A, the equilibrium limit for the yielding stress comes out directly from an infinitely delayed evolution map:

$$N_\infty(t_y) := \lim_{\gamma \to +\infty} N_\gamma(t_y); \tag{3.14}$$

because using N_γ evaluated for $t = t_y$ we obtain the value of the limit as $K_\infty E(t_y)$, we can define

$$|N_\infty(t_y)| = N_{ye}. \tag{3.15}$$

It is worth remarking that for every finite value of the mapping parameter γ we have $|N_\gamma(t_y)| > N_{ye}$, and only if γ goes to infinity we arrive at the equality (3.15). This limit and the sorting effect of the parameter γ emphasize the rate sensitivity of the yielding threshold (cf rel. (A.7) in Appendix A). In particular, by choosing $\gamma = 1$, the definitions of S and of S^* lead to the conclusion that $N_y = |N(t_y)|$ and the material behaves as a viscoplastic one only if the absolute value of the stress is greater than N_y. This threshold is crossed if the corresponding *relaxed stress*, computed and $K_\infty E(t)$, is greater or equal than N_{ye}.

Assuming a constant velocity deformation history, the constitutive law discussed above exhibits a quasi-bilinear stress evolution as shown in Fig. 2. An analogous behavior has been extensively discussed in Cernocky and Krempl (1979), in which the non-linearity of the constitutive relation was taken into account by means of the explicit dependence of the relaxation function G on the present value of the stress.

When unloading processes do not occur, eqns (3.9) and (3.10) are equivalent to describe the responses of a piecewise linear viscoelastic material with the following relaxation function

$$G(t) = \begin{cases} K_\infty + \bar{K} e^{-t\alpha} & \text{for} \quad 0 \leqslant t \leqslant t_y \\ H_\infty + \bar{H} e^{-t\beta} & \text{for} \quad t > t_y \end{cases} \tag{3.16}$$

where the measure of stress and strain are defined as in (3.5), (3.6), and this agrees with Krempl (1975) and Cernocky and Krempl (1979), who noted a rather similar behavior but in different contexts.

In the following we will consider deformation histories different from zero after a given finite instant, which will be taken as the zero time value.

Both the above-mentioned assumptions imply that $E(r) = 0$ for $-\infty < r < 0$, and the deformation histories appearing in (3.2) are such that $(t-r) \mapsto E(t-r)$, for $0 < r < t$.

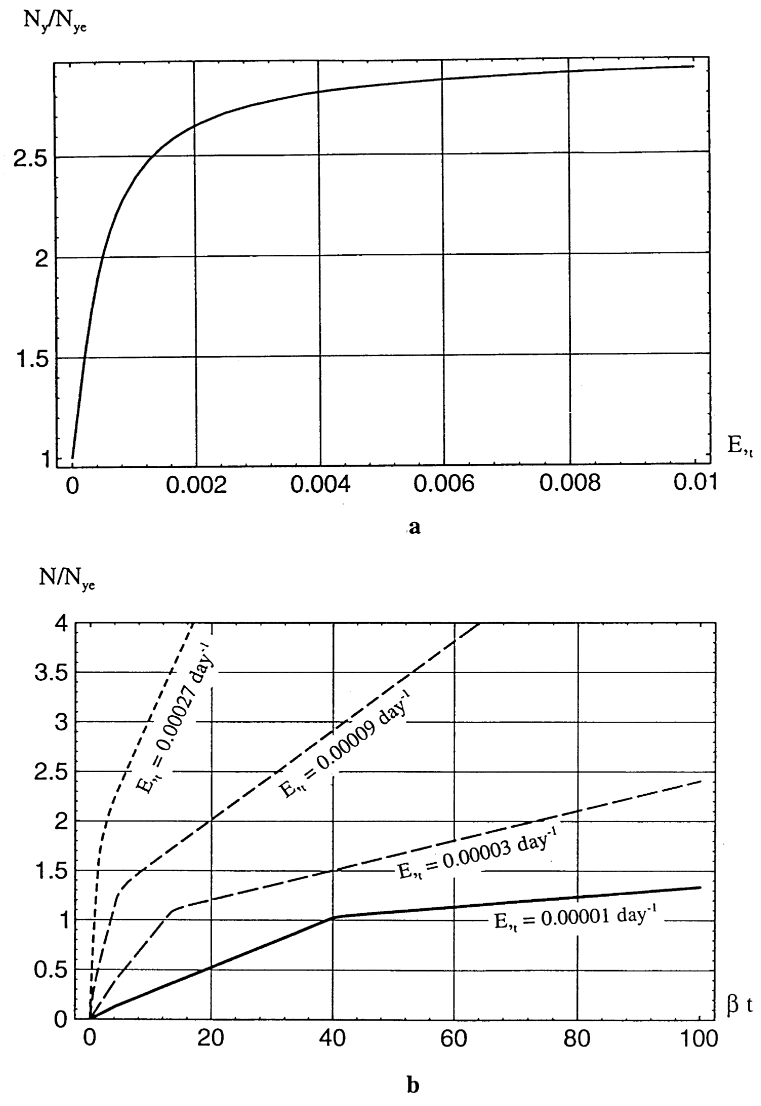

Fig. 2. Evolution of the stress under constant strain velocity: (a) yielding stress threshold as a function of the strain velocity; (b) non-dimensional stress as a function of the strain velocity. ($K = 15H_\infty$, $K_\infty = 5H_\infty$, $N_{ye} = 2H_\infty$, $\alpha = 2\beta$, $\beta = 10^{-3}\,\mathrm{day}^{-1}$).

4. Viscoelastic evolution

Starting from $t = 0$, we recall that if the first inequality of $(3.10)_1$ is verified for both springs, their behavior is linear and viscoelastic. Substituting the expression (3.3) of the relaxation function into the Boltzmann–Volterra eqn (3.2) and using the compatibility eqns (2.1), (2.2), the expressions of the normal stresses in the springs are:

$$N_1(t) = K\left[u(t) - \frac{B}{2}\phi(t)\right] - C_a \int_0^t e^{-(t-r)\alpha}\left[u(r) - \frac{B}{2}\phi(r)\right] dr, \tag{4.1}$$

$$N_2(t) = K\left[u(t) + \frac{B}{2}\phi(t)\right] - C_a \int_0^t e^{-(t-r)\alpha}\left[u(r) + \frac{B}{2}\phi(r)\right] dr, \tag{4.2}$$

after setting $C_a := \alpha\bar{K}$.

The equilibrium equations in terms of the unknown functions $t \mapsto \phi(t)$, $t \mapsto u(t)$ can be obtained by substitution of (4.1), (4.2) into eqn (2.4), i.e.

$$2Ku(t) - 2C_a \int_0^t e^{-(t-r)\alpha} u(r)\, dr = W$$

$$[KJ - Wh_G]\phi(t) - C_a J \int_0^t e^{-(t-r)\alpha}\phi(r)\, dr = Wh_G\phi_0 \tag{4.3}$$

where $J := B^2/2$. After straightforward calculations the solution of the system can be determined in the form:

$$u(t) = \frac{W}{2K_\infty}\left(1 - \frac{\bar{K}}{K} e^{-t\alpha K_\infty/K}\right), \tag{4.4}$$

$$\phi(t) = \frac{Wh_G\phi_0}{K_\infty J - Wh_G}\left\{1 - \frac{(K - K_\infty)J}{KJ - Wh_G} e^{-t\alpha\frac{K_\infty J - Wh_G}{KJ - Wh_G}}\right\}. \tag{4.5}$$

Until $S_2^*(t) < 0$, the evolution of the system is completely determined by eqns (4.4), (4.5), and in particular, if $K_\infty J - Wh_G > 0$, the exponent on the right-hand side of (4.5) is strictly negative, so that the limit of ϕ when $t \to +\infty$ becomes

$$\phi(+\infty) = \frac{Wh_G\phi_0}{K_\infty J - Wh_G}. \tag{4.6}$$

This means that the upper bound $\bar{W}$ of the external action introduced in (III) must verify the inequality

$$\bar{W} < \bar{W}_{ve}^{\infty}, \tag{4.7}$$

where we set:

$$\bar{W}_{ve}^{\infty} := \frac{K_\infty J}{h_G}. \tag{4.8}$$

By comparison between the expression (2.8) of the ultimate equilibrium rotation with (4.5) it is possible to check whether or not the system reaches the failure condition in a finite time t_{cr} in the viscoelastic phase. Introducing the non-dimensional ratios:

$$\omega := \frac{W}{\bar{W}_{ve}^{\infty}}, \quad \rho := \frac{K_\infty}{K}, \tag{4.9}$$

the limit condition arises if the following equation has a solution

$$\mathrm{e}^{-t_{cr}\alpha\frac{1-\omega}{\rho-\omega}} = \frac{\rho-\omega}{\rho-1}\left[\frac{\phi_0}{\phi_L} - \omega^{-1}\left(\frac{\phi_0}{\phi_L} - 1\right)\right]. \tag{4.10}$$

Thus, the sufficient condition to ensure that no failure occurs in the viscoelastic phase, is that the right-hand side must be strictly greater than one. After straightforward calculations this condition can be written as follows

$$\omega^2 - \omega\left[1 + \frac{1}{\rho}\left(1 - \frac{\phi_0}{\phi_L}\right)\right] + \frac{1}{\rho}\left(1 - \frac{\phi_0}{\phi_L}\right) > 0. \tag{4.11}$$

Since $\omega_1 = 1$, and $\omega_2 = (1/\rho)[1 - (\phi_0/\phi_L)]$ are the roots of the polynomial, the assumptions (II), (IV) ensures that $\omega_1 < \omega_2$, and inequality (4.11) is certainly verified if the external load is such that $W < \bar{W}_{ve}^{\infty}$. By (4.5), the value of the additional rotation at the beginning of the viscoelastic evolution can be determined

$$\phi(0) = \frac{Wh_G\phi_0}{KJ - Wh_G}, \tag{4.12}$$

which is positive and finite once $W < \bar{W}_E^0$, having introduced the Euler's critical load for the system:

$$\bar{W}_E^0 := \frac{KJ}{h_G}. \tag{4.13}$$

By assumption (V), $W < \bar{W}_E^0$ is verified once $W < \bar{W}_{ve}^{\infty}$.

So, the value $\bar{W}_{ve}^{\infty}$ may be taken as the minimum upper bound for the external load which ensures not only the uniqueness of the solution, but also that no failure occurs during the evolution of the system in the viscoelastic phase. However, this circumstance allows the possibility of yielding, which will be discussed in the next section.

For the sake of completeness, the values of the vertical displacement for $t = 0$, and for $t \to +\infty$, are computed:

$$u(0) = \frac{W}{2K}, \quad u(+\infty) = \frac{W}{2K_\infty}. \tag{4.14}$$

5. Viscoplastic evolution

The evolution of the system changes when the yielding occurs in one of the springs. Until N_1 and N_2 are under the yielding threshold, expression (4.3) holds, so that ϕ is monotonic increasing and, by eqns (2.5), (2.6), we have that $N_2 > N_1$. As it has already been noted in Section 3, the yielding parameter t_y is determined by the condition (3.13). For the particular problem under examination, we can replace $N_\infty(t_y)$ with $N_{2\infty}(t_y)$ in (3.13), which becomes

$$K_\infty\left[u(t_y)+\frac{B}{2}\phi(t_y)\right]=N_{ye}, \tag{5.1}$$

having introduced the compatibility eqn (2.3). In consideration of the expressions (4.4) and (4.5) for $t\mapsto u(t)$, $t\mapsto\phi(t)$, eqn (5.1) can be rewritten as follows:

$$1-(1-\rho)\,\mathrm{e}^{-t_y\alpha\rho}+\frac{\phi_0}{\phi_L(1-\omega)}\left[1-\frac{1-\rho}{1-\rho\omega}\mathrm{e}^{-t_y\alpha\rho\frac{1-\omega}{1-\rho\omega}}\right]=\frac{2\omega_{ye}}{\omega}, \tag{5.2}$$

where ω is defined by (4.9) and ω_{ye} is its value when $W=N_{ye}$. Exact solution of this equation can be obtained for $\phi_0=0$, i.e. for the perfect system. In this case expression (4.5) implies that $t\mapsto\phi(t)$ is identically zero, and (2.7) is verified as equality, i.e. $N_{2y}=N_2(t_y)=(W/2)$; the rate sensitivity property (3.24), ensures that $W/2$ is greater than the equilibrium yielding stress. In (5.2) only the first term on the left-hand side is different from zero, and we get

$$t_y=\ln\left\{(1-\rho)\frac{1}{\left(1-\frac{2\omega_{ye}}{\omega}\right)}\right\}^{\frac{1}{\rho\alpha}}, \tag{5.3}$$

so that the argument of the logarithm is meaningful if and only if

$$1<\frac{\omega}{2\omega_{ye}}<\frac{1}{\rho}. \tag{5.4}$$

The first inequality is certainly verified for the reason cited above, and this leads to the following definitions:

Definition 1: W is an *admissible* value for the external load if yielding is allowable with non-zero finite deformation velocity, i.e., the first of inequalities (5.4) is satisfied;

Definition 2: W is an *attainable* value for the external load if it is admissible and no ultimate equilibrium condition is reached.

Further, if any attainable load obeys the second inequality of (5.4), t_y is non zero, so that the viscoelastic phase is meaningful, and is passed within a finite time before the yielding. This fact will be discussed in the next section.

When ϕ_0 is different from zero, the value of the yield stress $N_2(t_y)$ can be obtained by expression (A.6) by taking $\gamma=1$, $t=t_y$, $t_1=0$, $N=N_2$, $E=E_2$, G as defined by (3.2), and E_2 given by the compatibility eqn (2.3), where the functions $t\mapsto u(t)$, $t\mapsto\phi(t)$ are given by (4.4) and (4.5), respectively. By setting:

$$\tilde{u}(t)=\int_0^t \mathrm{e}^{r\alpha}u(r)\,\mathrm{d}r=\frac{W}{2\alpha K_\infty}\mathrm{e}^{t\alpha}\{1-\mathrm{e}^{-t\alpha\rho}\}, \tag{5.5}$$

$$\tilde{\phi}(t) = \int_0^t e^{r\alpha}\phi(r)\,dr = \frac{\phi_0\omega}{\alpha(1-\omega)}e^{t\alpha}\left\{1-e^{-t\alpha\rho}\frac{1-\omega}{1-\rho\omega}\right\}, \tag{5.6}$$

the expression of $N_{2y} = N_2(t_y)$ becomes:

$$N_2(t_y) = K\left[u(t_y)+\frac{B}{2}\phi(t_y)\right]-C_a\,e^{-t_y\alpha}\left[\tilde{u}(t_y)+\frac{B}{2}\tilde{\phi}(t_y)\right]. \tag{5.7}$$

The substitution into eqn (5.7) of the expressions (4.4), (4.5), (5.5), (5.6) evaluated in t_y, yields, after some calculation, the following identity:

$$e^{-t_y\alpha\rho}\frac{1-\omega}{1-\rho\omega} = \frac{1-\rho\omega}{\omega(1-\rho)}\left[1-\frac{2N_2(t_y)-\omega}{K_\infty B}\frac{1-\omega}{\omega}\right]. \tag{5.8}$$

This expression is verified only if the right-hand side is a real number belonging to the interval [0, 1] and this circumstance occurs if the following inequalities are satisfied:

$$\frac{W}{2} \leqslant \frac{W}{2}\left(1+\frac{\phi_0/\phi_L}{1-\rho\omega}\right) \leqslant N_2(t_y) \leqslant \frac{W}{2}\left(1+\frac{\phi_0/\phi_L}{1-\omega}\right) \leqslant W, \tag{5.9}$$

and the first rough delimitation of $N_2(t_y)$ given by (2.7) and (2.9), evaluated for $t = t_y$ is refined. It is straightforward to check that the difference between the upper bound and the lower bound of $N_2(t_y)$ is strictly positive for $W < \bar{W}_E^0$, and, for every fixed value of the ratio ϕ_0/ϕ_L in agreement with (II), the bound difference increases as ρ decreases. The inequality between the second and the third member of (5.9) implies that the yield stress has to be sufficiently greater than $W/2$, and owing to (II), the latter inequality holds for every initial imperfection. The inequality under examination becomes:

$$\frac{\phi_0}{\phi_L} < 1-\omega, \tag{5.10}$$

which must be checked for $W < \bar{W}$ a-posteriori, once $\bar{W}$ introduced by (III) has been computed. To study the evolution after the yielding, the values of the functions ϕ and u for $t = t_y$ are requested, and their explicit expressions can be evaluated starting from (4.5), (4.4), respectively. When the yielding condition of (3.10) is satisfied, and if unloading processes do not occur, the present value of the stress N_2 may be determined by substitution of the expressions (3.16) for the relaxation function into the Boltzmann–Volterra equation; in view of the compatibility eqn (2.3) we have:

$$N_2(t) = K\left[u(t)+\frac{B}{2}\phi(t)\right]-C_b\int_0^t e^{-(t-r)\beta}\left[u(r)+\frac{B}{2}\phi(r)\right]dr$$
$$+e^{-(t-t_y)\alpha}[N_2(t_y)-KE_2(t_y)]+L(t), \tag{5.11}$$

after setting $C_b = \beta\bar{H}$ and:

$$L(t) := [\bar{H}(1-e^{-\beta(t-t_y)})-\bar{K}(1-e^{-\alpha(t-t_y)})]K_\infty^{-1}N_{ye}. \tag{5.12}$$

On the other hand, for the stress N_1 the first of conditions (3.10) is verified since $N_1(t) > 0$;

moreover, if $\phi_0 \neq 0$, it is straightforward to use (2.5), (2.7) and (5.0), to show that $N_1(t_y) < N_{ye}$. So, we can conclude that $N_1(t_y) < N_{1y}$, which implies $S_1^*(t_y) < 0$. The case $\phi_0 = 0$ follows a different demonstration line and is discussed in Appendix B.

Following the same procedure used to compute N_2, but replacing eqn (2.3) with eqn (2.2), we have:

$$N_1(t) = K\left[u(t)-\frac{B}{2}\phi(t)\right]-C_a\int_0^t \mathrm{e}^{-(t-r)\alpha}\left[u(r)-\frac{B}{2}\phi(r)\right]\mathrm{d}r+\mathrm{e}^{-(t-t_y)\alpha}[N_1(t_y)-KE_1(t_y)]. \tag{5.13}$$

The substitution of expression (5.11) and (5.13), into equilibrium eqns (2.4) leads to the following system:

$$2Ku(t)-\int_{t_y}^t (C_a\,\mathrm{e}^{-(t-r)\alpha}+C_b\,\mathrm{e}^{-(t-r)\beta})u(r)\,\mathrm{d}r+\frac{B}{2}\int_{t_y}^t (C_a\,\mathrm{e}^{-(t-r)\alpha}+C_b\,\mathrm{e}^{-(t-r)\beta})\phi(r)\,\mathrm{d}r = W(1+\lambda\,\mathrm{e}^{-(t-t_y)\alpha})-L(t) \tag{5.14}$$

$$\frac{B}{2}\int_{t_y}^t (C_a\,\mathrm{e}^{-(t-r)\alpha}-C_b\,\mathrm{e}^{-(t-r)\beta})u(r)\,\mathrm{d}r-\frac{J}{2}\int_{t_y}^t (C_a\,\mathrm{e}^{-(t-r)\alpha}+C_b\,\mathrm{e}^{-(t-r)\beta})\phi(r)\,\mathrm{d}r+(KJ-Wh_G)\phi(t) = W\phi_0(h_G+\mu\,\mathrm{e}^{-(t-t_y)\alpha})-\frac{B}{2}L(t) \tag{5.15}$$

after setting:

$$\lambda := 2\frac{C_a}{W}\mathrm{e}^{-t_y\alpha}\tilde{u}(t_y), \tag{5.16}$$

$$\mu := \alpha\frac{C_aJ}{W\phi_0h_G}\mathrm{e}^{-t_y\alpha}\tilde{\phi}(t_y); \tag{5.17}$$

λ and μ can be computed by means of expression (5.5), (5.6). As it was expected, coupling between the two unknown functions occurs in the viscoplastic range, and obviously this fact is due to the variation of both the equilibrium moduli and exponents from the viscoelastic to the viscoplastic phase.

Standard Laplace-transform technique can be applied to solve the system (5.14), (5.15). We denote with $s \mapsto \hat{U}(s)$ and $s \mapsto \hat{\Phi}(s)$ the transformed functions:

$$\hat{U}(s) = \mathscr{L}[u(t)] := \int_0^{+\infty} \mathrm{e}^{-s(t-t_y)}u(t)\,\mathrm{d}t \tag{5.18}$$

$$\hat{\Phi}(s) = \mathscr{L}[\phi(t)] := \int_0^{+\infty} \mathrm{e}^{-s(t-t_y)}\phi(t)\,\mathrm{d}t, \tag{5.19}$$

where the term e^{st_y} is the shifter which takes into account that in (5.14), (5.15), $t \mapsto u(t)$, $t \mapsto \phi(t)$ are defined for $t > t_y$. By taking the Laplace transform of both sides of (5.14), (5.15), we get

$$\left[2K-\left(\frac{C_a}{s+\alpha}+\frac{C_b}{s+\beta}\right)\right]\hat{U}(s)+\frac{B}{2}\left(\frac{C_a}{s+\alpha}+\frac{C_b}{s+\beta}\right)\hat{\Phi}(s) = W\left(\frac{1}{s}+\frac{\lambda}{s+\alpha}\right)-\hat{L}(s), \tag{5.20}$$

$$\frac{B}{2}\left(\frac{C_a}{s+\alpha}+\frac{C_b}{s+\beta}\right)\hat{U}(s)+\left\{\left[2K-\left(\frac{C_a}{s+\alpha}+\frac{C_b}{s+\beta}\right)\right]\frac{J}{2}-Wh_G\right\}\hat{\Phi}(s)$$
$$= W\phi_0 h_G\left(\frac{1}{s}+\frac{\mu}{s+\alpha}\right)-\frac{B}{2}\hat{L}(s) \tag{5.21}$$

where:

$$\hat{L}(s) := \int_0^{+\infty} \mathrm{e}^{-s(t-t_y)}L(t)\,\mathrm{d}t = \left(\frac{K_\infty - H_\infty}{s}+\frac{\bar{K}}{s+\alpha}-\frac{\bar{H}}{s+\beta}\right)K_\infty^{-1}N_{ye}. \tag{5.22}$$

The solution of the system can be obtained in terms of the transformed variable s:

$$\hat{U}(s) = \frac{U_0+U_1 s+U_2 s^2+U_3 s^3}{s(s+\alpha)^2(s+\beta)\Delta(s)}, \tag{5.23}$$

$$\hat{\Phi}(s) = \frac{\Phi_0+\Phi_1 s+\Phi_2 s^2+\Phi_3 s^3}{s(s+\alpha)^2(s+\beta)\Delta(s)} \tag{5.24}$$

where $\Delta(s)$ is the polynomial expression of the determinant for the linear system (5.20)–(5.21). The explicit expressions of $t \mapsto u(t)$, $t \mapsto \phi(t)$ are developed in Appendix C, by inversion of $s \mapsto \hat{U}(s)$ and $s \mapsto \hat{\Phi}(s)$. The function $\mathscr{L}$, which appears in definition (3.11), can be deduced by differentiation of (5.11) and (5.12) with respect to the time, to get:

$$\mathscr{L}(S_y, D_y, t) = \frac{\beta-\alpha}{\alpha}\mathrm{e}^{-\alpha(t-t_y)}\mathscr{G}_a(S_y, D_y)+\dot{L}(t)D_y. \tag{5.25}$$

6. Asymptotic behavior and conditions for attainability

In view of the final value theorem (Widder, 1941), the asymptotic behavior of $t \mapsto u(t)$, $t \mapsto \phi(t)$ can be studied directly by looking at the transformed functions. This allows to work out the expressions for both $u(+\infty)$ and $\phi(+\infty)$ as follows:

$$u(+\infty) = \lim_{s\to 0} s\hat{U}(s) = W\frac{(K_\infty+H_\infty)\dfrac{J}{2}-Wh_G+\left(\phi_0 h_G\dfrac{B}{2}-\dfrac{2N_{ye}}{W}J\right)(K_\infty-H_\infty)}{(K_\infty+H_\infty)\left(\dfrac{2K_\infty H_\infty}{K_\infty+H_\infty}J-Wh_G\right)}, \tag{6.1}$$

$$\phi(+\infty) = \lim_{s\to 0} s\hat{\Phi}(s) = W\frac{\phi_0 h_G(K_\infty + H_\infty) + \frac{B}{2}\left(1 - \frac{2N_{ye}}{W}\right)(K_\infty - H_\infty)}{(K_\infty + H_\infty)\left(\frac{2K_\infty H_\infty}{K_\infty + H_\infty}J - Wh_G\right)}. \tag{6.2}$$

The fact that $\phi(+\infty)$ is positive even for $\phi_0 \to 0^+$ can be explained noting that, although the yielding is reached at the same time in both the springs, at $t = t_y$ the evolution starts with different conditions in the two sides, because for one spring unloading inevitably occurs. Indeed, recalling relations (5.1) and (3.8), it follows that $S_1^*(t_y) = S_2^*(t_y) = 0$; moreover, the differentiation of eqns (2.5) and (2.6), under the hypothesis that W is constant, leads to $\dot{S}_1(t_y) < 0$ and $\dot{S}_2(t_y) > 0$; so, $(3.10)_1$ and $(3.10)_2$ are verified for the viscoelastic and viscoplastic parts, respectively. This explains why to consider $\phi_0 \to 0^+$ is not equivalent to take a perfect system from the beginning of the evolution. As cited previously, the analysis of the perfect system is carried out in detail in Appendix B.

Furthermore, examining (6.1) and (6.2) it appears that $u(+\infty)$ and $\phi(+\infty)$ assume finite values only if the external load W obeys the following inequality:

$$W < \bar{W}_{VK}^{\infty}, \tag{6.3}$$

where:

$$\bar{W}_{VK}^{\infty} := \frac{2K_\infty H_\infty}{K_\infty + H_\infty}\frac{J}{h_G}; \tag{6.4}$$

this value is exactly the Von Kármán critical load associated to the reduced modulus at infinity. By assumptions (IV), (V), (VI) is trivial to show that $\bar{W}_{VK}^{\infty} < \bar{W}_{ve}^{\infty}$; so, by taking into account inequality (4.7), we conclude that the greater external load for which there exists a unique equilibrated configuration for the system is exactly the Von Kármán critical load. As we pointed out previously, this result strongly depends on the circumstance that a constant force drives the system evolution.

It remains to investigate the conditions under which the ultimate equilibrium condition is reached, either asymptotically or in a finite time. The first case can be easily discussed by taking the expressions (2.8) and (6.2) for ϕ_u and $\phi(+\infty)$, respectively. It turns out that $\phi(+\infty) < \phi_u$ only if the external load satisfies the following inequality:

$$W < \bar{W}_U^{\infty}(\phi_0) \tag{6.5}$$

for any $\phi_0 > 0$ obeying (5.10), where:

$$\bar{W}_U^{\infty}(\phi_0) := \bar{W}_{SH}^{\infty}\left(1 - \frac{\phi_0}{\phi_L}\right) + N_{ye}\left(1 - \frac{H_\infty}{K_\infty}\right); \tag{6.6}$$

in (6.6) we indicated the Shanley critical load associated to the equilibrium tangent modulus as:

$$\bar{W}_{SH}^{\infty} = \frac{H_{\infty} J}{h_G}. \tag{6.7}$$

It is worth noting that because for $\phi_0 \to 0^+$ we have

$$\bar{W}_U^{\infty}(0^+) = \bar{W}_{SH}^{\infty} + N_{ye}\left(1 - \frac{H_{\infty}}{K_{\infty}}\right), \tag{6.8}$$

the same arguments used to explain the positiveness of $\phi(+\infty)$ justify by themselves the fact that the Shanley critical load is not reached when $\phi_0 \to 0^+$.

Moreover, the comparison between $\bar{W}_U^{\infty}(\phi_0)$ and $\bar{W}_{VK}^{\infty}$ tells us that the inequality:

$$\bar{W}_U^{\infty}(\phi_0) < \bar{W}_{VK}^{\infty}, \tag{6.9}$$

holds for any $\phi_0 > 0$ obeying (II), only if

$$\bar{W}_{VK}^{\infty} \cdot \left(1 + \frac{\phi_0}{\phi_L} \frac{K_{\infty} + H_{\infty}}{K_{\infty} - H_{\infty}}\right) > 2N_{ye}; \tag{6.10}$$

this inequality is certainly verified by $\bar{W}_{VK}^{\infty}$ is admissible. So, collecting (2.9), (4.7), (6.4) and (6.5), we can conclude that if the external load obeys to the inequalities

$$\bar{W}_{ve}^{\infty} > \bar{W}_{VK}^{\infty} > \bar{W}_U^{\infty}(\phi_0) > W > 2N_{ye} \tag{6.11}$$

uniqueness of the solution of the equilibrium problem is ensured, no failure occurs in the viscoelastic phase and no asymptotic ultimate equilibrium condition arises.

The monotonicity of $t \mapsto \phi(t)$ implies that (6.11) is also a sufficient condition for the non-existence of a finite critical time during the system evolution in the viscoelastic range: so, a sufficient condition for W to be attainable according to (definition 2) has been determined.

It is noticeable that the condition imposed by (6.11) is meaningful only if $\bar{W}_U^{\infty}(\phi_0)$ is admissible, and a sufficient condition ensuring this property for any positive value of ϕ_0 obeying (II) is that:

$$\bar{W}_{SH}^{\infty} > N_{ye}\left(1 + \frac{H_{\infty}}{K_{\infty}}\right). \tag{6.12}$$

Since constitutive assumption (IV) holds true, the latter inequality does not guarantee the admissibility of $\bar{W}_{SH}^{\infty}$, and this is in agreement with what we already remarked about the fact that the imperfection amplitude evolves under constant load.

Further, by eqns (6.6) and (6.8) it is possible to prove that $W_U^{\infty}(\phi_0) > W_{SH}^{\infty}$, for every value of $\phi_0 > 0$ obeying (II). In fact we need only to prove that:

$$\frac{\phi_0}{\phi_L} < \frac{N_{ye}}{\bar{W}_{SH}^{\infty}}\left(1 - \frac{H_{\infty}}{K_{\infty}}\right) < \frac{K_{\infty} - H_{\infty}}{K_{\infty} + H_{\infty}} < 1, \tag{6.13}$$

and this can be easily obtained by virtue of (6.12), assumption (IV), and inequality (6.10). The analysis carried out before leads to the conclusion that the minimum upper bound of the load introduced in (III) is given by $\bar{W}_U^{\infty}(\phi_0)$.

At the beginning of this section, relations (5.4) were proved to hold for the perfect system. For sake of completeness, choosing $W = \bar{W}_U^{\infty}(\phi_0)$ we verify whether the right inequality holds when any positive ϕ_0 obeying to (II) is present. We have:

$$\bar{W}_U^{\infty}(\phi_0) < 2N_{ye}\rho, \tag{6.14}$$

which takes the form:

$$1-\frac{\phi_0}{\phi_L} < 2\omega_{ye}\frac{1}{H_\infty}\left(K-\frac{K_\infty-H_\infty}{2}\right) \tag{6.15}$$

Taking into account (6.12) and noting that a sufficient condition for this inequality to hold can be determined for $\phi_0 \to 0^+$, we get:

$$\frac{H_\infty}{2K-K_\infty+H_\infty} < \omega_{ye} < \frac{H_\infty}{K_\infty+H_\infty}. \tag{6.16}$$

It is worth noting that the left-hand side is strictly less than 1/2 and its denominator is positive by virtue of (VI). The range determined for ω_{ye} ensures that no instantaneous yielding can arise because $t_y \neq 0$.

So far, we pointed out that by the fourth inequality of (5.9), the upper bound of the yielding value $N_2(t_y)$ is meaningful only if (5.10) is satisfied for every attainable load. This condition must be verified when W equals $\bar{W}_U^{\infty}(\phi_0)$ for any value of $\phi_0 > 0$ obeying (II). The substitution of expression (6.6) in (5.10) after some easy calculations leads to the following inequality:

$$\frac{\phi_0}{\phi_L} < 1-\omega_{ye}. \tag{6.17}$$

The substitution in this relation of the upper bound of ρ_{ye} determined by the right-hand side of inequality (6.6), yields

$$\frac{\phi_0}{\phi_L} < \frac{K_\infty}{K_\infty+H_\infty}, \tag{6.18}$$

which is certainly verified if (II) holds.

7. Concluding remarks

As it was noted in Shanley (1947), for the classical elastic–plastic case bifurcation phenomena can arise once the external load reaches one of three different values, depending whether the sway of the system occurs when the yield threshold has been reached in none, one or both the springs. Moreover, the loss of uniqueness appears for any value of the load parameter greater than the Shanley's critical load and smaller than the Euler's critical one, being the Von Kármán's value in between. Only if the rotation of the system does evolve forced by an increasing external load, with magnitude large enough to have positive increment of both the spring normal stresses, the bifurcation of the equilibrium takes place at the Shanley's critical load. However, in the viscoplastic case this is no longer true.

In agreement with the Von Kármán's statement, inequality (6.4) leads to the first result, i.e. bifurcated configurations can occur under constant load once the external force equals $\bar{W}^{\infty}_{VK}$.

On the other hand, if $W < \bar{W}^{\infty}_{U}(\phi_0) < \bar{W}^{\infty}_{VK}$ the exclusion of asymptotic failure is also sufficient to say that bifurcation phenomena do not arise, and this is verified once $\bar{W}^{\infty}_{VK}$ is admissible. The chain of inequalities (6.13) proves that if $\bar{W}^{\infty}_{U}(\phi_0)$ is admissible as well, it is also the minimum value among all the loading parameters characterizing the phases of the system evolution, i.e. $\bar{W} = \bar{W}^{\infty}_{U}(\phi_0)$ and every load which verifies (III) is attainable.

A sufficient condition for the admissibility of $\bar{W}^{\infty}_{U}(\phi_0)$ is that the Von Kármán's critical load $\bar{W}^{\infty}_{VK}$ be admissible itself, and this is true if the Shanley's load $\bar{W}^{\infty}_{SH}$ is large enough according to (6.12). This condition is certainly verified if also $\bar{W}^{\infty}_{SH}$ is admissible, which is false if the system has an initial imperfection. Only if $\phi_0 = 0$ the Shanley's load turns out to be admissible, even when a constant external load drives the evolution (see Appendix B).

Nonetheless, the Shanley critical load summed up with the above-mentioned positive term plays a key role on the characterization of the imperfect system equilibrium in the viscoplastic phase. The obtained results do depend only on the moduli and yielding stress at equilibrium, so that any expression for the relaxation surface consistent with the constitutive assumptions and such that it converges to (3.12) under extreme retardation, leads to the same conclusion.

Appendix A: Time rescaling of the process

In order to obtain a rescaled process, we fix an interval $[t_1, t_2]$ and we take a real $\gamma > 0$ to construct the map τ_γ defined as follows:

$$\tau_\gamma := \begin{cases} t+(1-\gamma)(t_2-t_1) & \text{for} \quad t < t_1 \\ \gamma t+(1-\gamma)t_2 & \text{for} \quad t_1 \leqslant t < t_2 \\ t & \text{for} \quad t \geqslant t_2 \end{cases}. \tag{A.1}$$

The function τ_γ maps $[t_1, t_2]$ into the interval $[t_2-\gamma(t_2-t_1), t_2]$ and it acts on $[t_1, t_2]$ either with a uniform contraction if $\gamma < 1$, or with a uniform dilatation if $\gamma > 1$. Further, we say either that the following function:

$$D_\gamma(\tau_\gamma(t)) := D(t), \tag{A.2}$$

is the γ-acceleration of D in $[t_1, t_2]$ if $\gamma < 1$, or the γ-retardation if $\gamma > 1$. For $t \geqslant t_2$ and for all positive γ we have:

$$D_\gamma(t) = D(t), \tag{A.3}$$

and for $t < t_2$,

$$D(t) := \begin{cases} D(t-(1-\gamma)(t_2-t_1)) & \text{for} \quad t < t_2-\gamma(t_2-t_1) \\ D(t_2-\gamma^{-1}(t_2-t_1)) & \text{for} \quad t \geqslant t_2-\gamma(t_2-t_1) \end{cases}. \tag{A.4}$$

The decreasing monotonicity of the relaxation function G ensures that the same property holds for the function:

$$\gamma \mapsto S_\gamma(t) := K \cdot D_\gamma(t) + \int_{-\infty}^{t} \dot{G}(t-r) \cdot D_\gamma(r)\,\mathrm{d}r, \tag{A.5}$$

obtained by (3.1) replacing S and D with S_γ, D_γ. By inserting into (A.5) the expression (A.4) and (3.2), a useful form of the latter equation can be deduced after integration by parts:

$$S_\gamma(t) = K_\infty D(t) + \bar{K} \int_{t_1}^{t} \mathrm{e}^{-\gamma\alpha(t-r)}\,\mathrm{d}D(r) + \bar{K}\mathrm{e}^{-\gamma\alpha(t-t_1)}\left(-D(t_1) + \int_{-\infty}^{t_1} \mathrm{e}^{-\gamma\alpha(t_1-r)}\,\mathrm{d}D(r)\right). \tag{A.6}$$

If we consider two different values $\gamma_1 < \gamma_2$, eqn (3.17) and the monotonicity of G lead to the inequality:

$$S_{\gamma 1}(t) > S_{\gamma 2}(t). \tag{A.7}$$

In view of (A.4) and by substituting eqn (A.6) into (3.4), the value of the relaxation surface $\mathscr{G}_a$ at $\{S_\gamma(t), D_\gamma(t)\}$ takes the form:

$$\mathscr{G}_a(S_\gamma(t), D_\gamma(t)) = -\alpha\bar{K}\left[\int_{t_1}^{t} \mathrm{e}^{-\gamma\alpha(t-r)}\,\mathrm{d}D(r) + \mathrm{e}^{-\gamma\alpha(t-t_1)}\left(-D(t_1) + \int_{-\infty}^{t_1} \mathrm{e}^{-\gamma\alpha(t_1-r)}\,\mathrm{d}D(r)\right)\right]. \tag{A.8}$$

If t_p denotes the first instant in which the loading condition of (3.10) is verified, and if no unloading occur in the interval $[t_p, t]$, $t > t_p$ fixed, by taking $t_1 = t_p$ the relaxation surface is given by eqn (3.11), and its value in $\{S_\gamma(t), D_\gamma(t)\}$ is:

$$\mathscr{G}_b(S_\gamma(t), D_\gamma(t)) = -\beta\bar{H}\left[\int_{t_p}^{t} \mathrm{e}^{-\gamma\beta(t-r)}\,\mathrm{d}D(r) + \bar{H}^{-1}\bar{K}\mathrm{e}^{-\gamma\alpha(t-t_p)}\left(-D(t_p) + \int_{-\infty}^{t_p} \mathrm{e}^{-\gamma\alpha(t_p-r)}\,\mathrm{d}D(r)\right)\right]$$
$$-\frac{\beta-\alpha}{\alpha}\mathrm{e}^{-\alpha\gamma(t-t_p)}\,\mathscr{G}_a(S(t_p), D(t_p)), -\dot{L}(\gamma t) \cdot D(t_p) \tag{A.9}$$

The functions $\gamma \mapsto \mathscr{G}_a(S_\gamma(t), D_\gamma(t))$ and $\gamma \mapsto \mathscr{G}_b(S_\gamma(t), D_\gamma(t))$ are strictly monotonic decreasing, and converge to the null value in the limit for $\gamma \to +\infty$.

Appendix B: Viscoplastic bifurcation of the perfect system

As we already noted at the beginning of Section 5, if $\phi_0 = 0$ the function $t \mapsto \phi(t)$ is identically zero in both the elastic and the viscoelastic phases and $N_{1y} = N_{2y} = (W/2)$ leading so, by definition (3.8), to $S_1^*(t_y) = S_2^*(t_y) = 0$. Since the load is constant, by differentiation of eqns (2.5), (2.6), we have $\dot{S}_1(ty) = \dot{S}_2(ty) = 0$ so that $(3.10)_2$ is verified. So, N_2 is given by (5.11) and N_1 takes the following form:

$$N_1(t) = K\left[u(t) - \frac{B}{2}\phi(t)\right] - C_b \int_{t_y}^{t} \mathrm{e}^{-(t-r)\beta}\left[u(r) - \frac{B}{2}\phi(r)\right]\mathrm{d}r$$
$$+\mathrm{e}^{-(t-t_y)\alpha}(N_1(t_y) - KE_1(t_y)) + L(t) \tag{B.1}$$

Since $E_1(t_y) = E_2(t_y) = u(t_y)$ the third term is equal for both the expressions of N_1 and N_2, so that the equilibrium eqns (2.4) become:

$$Ku(t) - C_b \int_{t_y}^{t} \mathrm{e}^{-(t-r)\beta} u(r)\,\mathrm{d}r = [K + \bar{K}(1 - \mathrm{e}^{-(t-t_y)\alpha}) - \bar{H}(1 - \mathrm{e}^{-(t-t_y)\beta})] K_\infty^{-1} N_{ye}, \tag{B.2}$$

$$(KJ - Wh_G)\phi(t) + J \int_{t}^{t} C_b \mathrm{e}^{-(t-r)\beta} \phi(r)\,\mathrm{d}r = 0. \tag{B.3}$$

The first equation leads directly to the expression of $u(t)$:

$$u(t) = [K_\infty \mathrm{e}^{-(t-t_y)\beta} + (\bar{K} + H_\infty)(1 - \mathrm{e}^{-(t-t_y)\beta})] K_\infty^{-2} N_{ye} + \frac{(\alpha - \beta)\bar{K}}{\beta K_\infty - \alpha K} \mathrm{e}^{-(t-t_y)\alpha}, \tag{B.4}$$

while the second one is an eigenvalue problem. It is worth noting that if (VI) holds $u(+\infty) > u(t_y)$, and this is in agreement with the result of (B.4), namely $u(+\infty) = (H_\infty + \bar{K}) K_\infty^{-2} N_{ye}$.

Following the time rescaling procedure introduced in Appendix A, either by taking the γ-acceleration or the γ-retardation of ϕ in $(t_y, t]$, and integrating by parts, (B.3) becomes:

$$(H_\infty J - Wh_G)\phi(t) + \bar{H}J \int_{t}^{t} C_b \, \mathrm{e}^{-\gamma(t-r)\beta} \phi(r)\,\mathrm{d}r + \bar{H}J \, \mathrm{e}^{-\gamma(t-t_y)\beta} \phi(t_y) = 0. \tag{B.5}$$

The extreme acceleration and the extreme retardation of ϕ yield the following expressions:

$$(KJ - Wh_G)\phi(t) = 0, \quad (H_\infty J - Wh_G)\phi(t) = 0, \tag{B.6}$$

whose solution are the bifurcation load parameters $\bar{W}_E^0$, $\bar{W}_{SH}^\infty$, although the first one is excluded by inequality (4.7). It follows that, if the system is perfect at the beginning of its evolution, and no small disturbances arise neither in the viscoelastic phase nor in the viscoplastic phase, the Shanley's critical load is admissible and bifurcation may occur for $W = \bar{W}_{SH}^\infty$.

The same result can be obtained directly by solving eqn (B.3), which admits the eigensolution:

$$\phi(t) = A \, \mathrm{e}^{-(t-t_y)\beta \frac{H_\infty J - Wh_G}{KJ - Wh_G}}, \tag{B.7}$$

where A is an arbitrary constant; indeed, we realize that this function can be of bounded variation on $(t_y, +\infty)$, only if $W \leqslant \bar{W}_{SH}^\infty$, proving the thesis.

Appendix C: Inversion of $s \mapsto \hat{U}(s)$ and $s \mapsto \hat{\Phi}(s)$

Here we consider the determinant of the system of equations (5.20) and (5.21):

$$\Delta(s) = \frac{\Delta_0 + \Delta_1 s + \Delta_2 s^2}{(s + \alpha)(s + \beta)}, \tag{C.1}$$

where

$$\Delta_0 := \alpha\beta(K_\infty + H_\infty)\left(\frac{2K_\infty H_\infty}{K_\infty + H_\infty}J - Wh_G\right), \tag{C.2}$$

$$\Delta_1 := 2K(\alpha+\beta)\left(\frac{\alpha K_\infty + \beta H_\infty}{\alpha+\beta}J - Wh_G\right), \tag{C.3}$$

$$\Delta_2 := 2K(KJ - Wh_G). \tag{C.4}$$

The solution of the system can be written as:

$$\hat{U}(s) = \frac{U_0 + U_1 s + U_2 s^2 + U_3 s^3}{s(s+\alpha)^2(s+\beta)\Delta(s)}, \tag{C.5}$$

where:

$$U_0 = \alpha^2\beta\left[(K_\infty + H_\infty)\frac{J}{2} - Wh_G + (K_\infty - H_\infty)\left(\phi_0 h_G \frac{B}{2} - \frac{2N_{ye}}{W}J\right)\right], \tag{C.6}$$

$$U_1 = \alpha[\beta(2+\lambda)+\alpha](KJ - Wh_G) + (K_\infty - H_\infty)\left(\phi_0 h_G \frac{B}{2}\beta(1+\mu) - \frac{2N_{ye}}{W}J\frac{K_\infty + H_\infty}{K_\infty}\right) + \phi_0 h_G \frac{B}{2}(\alpha K - \beta H) \tag{C.7}$$

$$U_2 = \alpha(KJ - Wh_G) - (\alpha\bar{K} - \beta\bar{H})\left(\phi_0 h_G \frac{B}{2}(1+\mu) - \frac{2N_{ye}}{W}J\frac{K_\infty + H_\infty}{K_\infty}\right) + (1+\lambda)\left[\frac{J}{2}(\alpha K_\infty + \beta H_\infty) - Wh_G(\alpha+\beta)\right] \tag{C.8}$$

$$U_3 = (1+\lambda)(KJ - Wh_G). \tag{C.9}$$

In the same manner:

$$\hat{\Phi}(s) = \frac{\Phi_0 + \Phi_1 s + \Phi_2 s^2 + \Phi_3 s^3}{s(s+\alpha)^2(s+\beta)\Delta(s)} \tag{C.10}$$

where:

$$\Phi_0 = \alpha^2\beta\left[(K_\infty + H_\infty)\phi_0 h_G + (K_\infty - H_\infty)\left(1 - \frac{2N_{ye}}{W}\right)\frac{B}{2}\right], \tag{C.11}$$

$$\Phi_1 = \alpha\phi_0 h_G[2K(\alpha+\beta) - (\alpha\bar{K} + \beta\bar{H}) - \beta(1+\mu)(K_\infty - H_\infty)] - \alpha\frac{B}{2}\left[(\alpha\bar{K} - \beta\bar{H}) - \beta(K_\infty - H_\infty)\left(1 + \lambda - \frac{2N_{ye}}{W}\frac{K_\infty + H_\infty}{K_\infty}\right)\right], \tag{C.12}$$

$$\Phi_2 = \phi_0 h_G\{2K\alpha + (1+\mu)[\alpha(K+K_\infty) + \beta(K+H_\infty)]\} - \frac{B}{2}(\alpha\bar{K} - \beta\bar{H})\left(1 + \lambda - \frac{2N_{ye}}{W}\frac{\bar{K}}{K_\infty}\right) \quad (C.13)$$

$$\Phi_3 = 2K(1+\mu)\phi_0 h_G. \quad (C.14)$$

The functions (C.5) and (C.10) give the solution of the evolution problem in terms of the transformed variable s; in order to perform an inverse Laplace transform, the algebraic form of the function must be resolved in a sum of factored terms.

Examining the case of the rotation (but the same holds for the displacement too), by substitution of (C.1) in (C.10) we have:

$$\hat{\Phi}(s) = \frac{W}{\alpha}\frac{\Phi_0 + \Phi_1 s + \Phi_2 s^2 + \Phi_3 s^3}{(\alpha + s)\cdot(s - s_1)\cdot(s - s_2)}, \quad (C.15)$$

where we indicated with s_1 and s_2 the two real solutions of the equation:

$$\Delta_0 + \Delta_1 s + \Delta_2 s^2 = 0. \quad (C.16)$$

In order to perform analytically the inverse transform we are seeking for a form of the type:

$$\hat{\Phi}(s) = \frac{W}{\alpha}\left[\frac{A}{s} + \frac{B}{s+\alpha} + \frac{C}{s-s_1} + \frac{D}{s-s_2}\right]. \quad (C.17)$$

The unknown coefficients A to D are to be determined imposing the condition that the difference (C.15)–(C.17) must vanish for all possible s values. Expanding the difference and solving the linear system in the four unknown coefficients we have finally:

$$A = \frac{\Phi_0}{\alpha s_1 s_2}, \quad B = \frac{\Phi_0 + \Phi_1\alpha - \Phi_2\alpha^2 + \Phi_3\alpha^3}{\alpha\cdot(\alpha + s_1)\cdot(\alpha + s_2)}, \quad (C.18a)$$

$$C = \frac{\Phi_0 + \Phi_1 s_1 + \Phi_2 s_1^2 + \Phi_3 s_1^3}{s_1\cdot(\alpha + s_1)\cdot(s_1 - s_2)}, \quad D = \frac{\Phi_0 + \Phi_1 s_2 + \Phi_2 s_2^2 + \Phi_3 s_2^3}{s_2\cdot(\alpha + s_2)\cdot(s_2 - s_1)}. \quad (C.18b)$$

Thus, transforming back, the time evolution of the rotation can be expressed as a function of the time variable and the four coefficients reported above; we have:

$$\mathscr{L}^{-1}[\mathrm{e}^{st_y}\hat{\Phi}(s)] = W(A + B\,\mathrm{e}^{-\alpha\tau} + C\,\mathrm{e}^{s_1\tau} + D\,\mathrm{e}^{s_2\tau}) = \phi(\tau), \quad (C.19)$$

where:

$$\tau = t - t_y. \quad (C.20)$$

References

Basquin, O.H., 1924. The tangent modulus and the strength of steel columns in tests. Natl Bur. Standards Technol. Papers 263, 381–442.

Bazant, Z.P., Cedolin, L., 1991. Stability of Structures. Oxford University Press, New York.

Burland, P., Viggiani, C., 1994. Osservazioni sul comportamento della torre di Pisa. Rivista Italiana di Geotecnica 3, 179–200.
Cernocky, E.P., Krempl, E., 1979. A non-linear uniaxial integral constitutive equation incorporating rate effects, creep and relaxation. Int. J. of Non-linear Mech. 14, 183–203.
Chaboche, J.L., 1993. Cyclic viscoplastic constitutive equations: Part I: A thermodynamically consistent formulation. J. Appl. Mech. 60, 813–821.
Chaboche, J.L., Rousselier, G., 1983. On the plastic and viscoplastic constitutive equations: Part I: Rules developed with internal variable concept. J. Pressure Vessel Technol. 105, 153–158.
Cheney, J.A., Abghari, A., Kutter, B.L., 1991. Stability of leaning towers. ASCE J. of Geotechnical Engnrg 117 (2), 297–318.
Cimetière, A., Léger, A., 1996. Some problems about elastic plastic post buckling. Int. J. Sol. Struct. 33, 1519–1533.
Coleman, B.D., Noll, W., 1961. Foundations of linear viscoelasticity. Rev. Mod. Phys. 33, 239–261.
Considère, F., 1891. Resistance des pièces comprimées. Congr. Int. Procedés de Constr., Paris 3, 371.
Cristescu, N., Suliciu, I., 1982. Viscoplasticity. Martinus Nijhoff, The Hague.
Del Piero, G., Deseri, L., 1995. Monotonic, completely monotonic and exponential relaxation functions in linear viscoelasticity. Quart. of Appl. Math. 53, 273–300.
Del Piero, G., Deseri, L., 1997. On the concept of state and free energy in linear viscoelasticity. Arch. for Rat. Mech. and An. 138, 1–35.
Dill, E.H., 1975. Simple materials with fading memory. In: Eringen, A.C. (Ed.), Continuum Physics, vol. 2, Continuum Mechanics and Simple-Substance Bodies. Academic Press.
Duberg, J.E., 1962. Inelastic buckling. In: Flügge, W. (Ed.), Handbook of Engineering Mechanics, Chap. 52, 1–9. McGraw-Hill, New York.
Engesser, F., 1889. Über Knickfestigkeit Gerader Stäbe. Z. Archit. und Ing. Wesen 35, 455.
Fabrizio, M., Giorgi, C., Morro, A., 1994. Free energies and dissipation properties for systems with memory. Arch. for Rat. Mech. and Anal. 125, 341–373.
Gurtin, M.E., Sternberg, E., 1962. On the linear theory of viscoelasticity. Tech. Report No. 6, Contract No. 562(30), Brown University, Providence, RI.
Gurtin, M.E., Williams, W.O., Suliciu, I., 1980. On rate-type constitutive equations and the free energy of viscoelastic and viscoplastic materials. Int. J. Sol. Struct. 16, 607–617.
Hambly, E.C., 1985. Soil buckling and leaning instability of tall towers. Struct. Engrg 63, 37, 77–85.
Haupt, P., 1992. Thermodynamics of solids. IN: Nonequilibrium Thermodynamics with Application to Solids, C.I.S.M. Course No. 59, Udine, 28 September–2 October.
Haupt, P., 1993. On the mathematical modeling of material behavior in continuum mechanics. Acta Mechanica 100, 129–154.
Heyman, J., 1992. Leaning towers. Meccanica 27, 153–159.
Hill, R., 1960. A general theory of inelastic column failure—I and II. J. of the Mech. and Phys. of Solids 8, 105–111, 112–119.
Hoff, N.J., 1954. Buckling and Stability. J. Royal. Aeron. Soc., 58, 1–52.
Hoff, N.J., 1956. Creep buckling. Aero. Quart. 7, 1–20.
Hoff, N.J., 1958. A survey of the theories of creep buckling. Proc. 3rd US Natl Congr. Appl. Mech. Providence, RI, pp. 29–49.
Hutchinson, J.W., 1974. Plastic buckling. Adv. in Appl. Mech. 14.
Kempner, J., 1962. Viscoelastic buckling. In: Flügge, W. (Ed.), Handbook of Engineering Mechanics 54, 1–16. McGraw-Hill, New York.
Kratochvil, J., Dillon, O.W., 1969. Thermodynamics of elastic–plastic materials as a theory with internal state variables. J. Appl. Phys. 40, 3207–3218.
Krempl, E., 1975. On the interaction of rate and history dependence in structural metals. Acta Mechanica 22, 53.
Lancellotta, R., 1993. Stability of a rigid column with non-linear restraint. Géotechnique 43 (2), 331–332.
Libove, J., 1952. Creep buckling of columns. J. Aeron. Sci. 19, 459–467.
Lubliner, J., 1973. On the structure of the rate equations of materials with internal variables. Acta Mechanica 17, 109–119.
Malinin, N.N., Khandjinsky, G.M., 1972. Theory of creep with anisotropic hardening. Int. J. Mech. Sci. 14, 235–246.
Naghdi, P.M., Murch, S.A., 1963. On the mechanical behaviour of viscoelastic–plastic solids. J. of Appl. Mech. 321–328.
Nova, R., Montrasio, L., 1995. Un'analisi di stabilità del campanile di Pisa. Rivista Italiana di Geotecnica 2, 83–93.

Odqvist, F.K.G., 1966. Mathematical Theory of Creep and Creep Rupture. Clarendon Press, Oxford.

Oka, F., Adachi, T., Mimura, M., 1988. Elasto-viscoplastic constitutive models for clays. Int. Conf. on Rheology and Soil Mechanics. Coventry, U.K.

Perzyna, P., 1963. The constitutive equations for rate sensitive plastic materials. Quart. of Appl. Math. 20, 321–332.

Perzyna, P., 1966. Fundamental problems in viscoplasticity. Adv. in Appl. Mech. 9, 243–377.

Rabotnov, Y.N., 1969. Creep Problems in Structural Members. North-Holland, Amsterdam, London.

Rabotnov, Y.N., Shesterikov, S.A., 1957. Creep stability of columns and plates. J. of Mech. and Phys. of Solids 6.

Rzhanitsyn, A.R., 1968. Theory of Creep. (in Russian). Moskva, Strojizdat.

Sewell, M.J., 1971. A survey of plastic buckling. Struct. Mech. Study No. 6 5, University of Waterloo, Ontario, Canada.

Shanley, F.R., 1947. Inelastic column theory. Int. J. of Aero. Sci. 14, 261–267.

Triantafyllidis, N., 1983. On the bifurcation and postbifurcation analysis of elastic–plastic solids under general prebifurcation conditions. J. of Mech. and Phys. of Solids 31 (6), 499–510.

Tsakmakis, C.H., 1996. An analysis of rate- and material parameter-dependent limiting cases in viscoplasticity laws. Int. J. Sol. Struct. 33 (2), 149–166.

Vinogradov, A.M., 1985. Nonlinear effects in creep buckling analysis of columns. J. of Eng. Mech. 111 (6), 757–767.

Von Kármán, Th., 1910. Untersuchungen über Knickfestigkeit. Mitt. Forschungsarb. V.D.I., 81; also in Collected Works 1, 90–140. Butterworth, London.

Widder, D.V., 1941. The Laplace Transform. Princeton University Press.